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ON ASYMPTOTIC INTEGRATION OF THE EQUATIONS OF MOTION OF MECHANICAL SYSTEMS SUBJECTED TO RAPIDLY OSCILLATING FORCES*

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An algorithm for the direct expansion of solutions of the Cauchy problem in a small parameter in a finite time interval is proposed in the development of the idea in the author's paper /1/ for systems of differential equations describing the motion of mechanical systems subjected to rapidly oscillating forces.

We consider a mechanical system whose motion is described by the vector differential equation

$$A(q)q^{\cdot\cdot} + B(q)q^{\cdot} = F(t, q) + \omega\Phi(t, q, \tau) \qquad (1)$$

where $q = (q^1, \ldots, q^n)$ is the generalized coordinate vector, the dot denotes differentiation with respect to time t, A is a positive-definite matrix of the inertial forces, B is the matrix of the dissipative forces, $\omega\Phi$ are large amplitude oscillating forces ($\omega \gg 1, \tau = \omega t$). For simplicity we will consider Φ to be a trigonometric polynomial in τ of period 2π, with zero mean in τ. Let the following initial conditions be given

$$q(0) = \alpha, \; q^{\cdot}(0) = \beta \qquad (2)$$

We will seek the approximate solution of the Cauchy problem (1) and (2) in the form

$$q^* = u_0(t) + \omega^{-1}[u_1(t) + v_1(t, \tau)] + \ldots + \omega^{-s}[u_s(t) + v_s(t, \tau)] + \ldots \qquad (3)$$

where $v_i(t, \tau)$ are periodic functions of τ of period 2π with zero mean value. The sum $u_0 + \omega^{-1}u_1 + \ldots$ is the smooth motion component while $\omega^{-1}v_1 + \omega^{-2}v_2 + \ldots$ is the vibrational component. We have

$$A(q^*) = A^\circ + \omega^{-1}A_q^\circ(u_1 + v_1) +$$
$$\omega^{-2}A_q^\circ(u_2 + v_2) + {}^1/_2 A_{qq}^\circ(u_1 + v_1)^2 + \cdots$$
$$(A^\circ = A(u_0), \; A_q^\circ = A_q(u_0), \ldots)$$

Analogous expressions hold for $B(q^*), F(t, q^*), \ldots$.

We obtain from the initial conditions (2), formulas (3) and the result of differentiating (3) with respect to t

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$$u_0(0) + \omega^{-1}[u_1(0) + v_1(0,0)] + \omega^{-2}[u_2(0) + v_2(0,0)] + \ldots = \alpha$$
$$u_0^{\cdot}(0) + \partial v_1(0,0)/\partial\tau + \omega^{-1}[u_1^{\cdot}(0) + v_1^{\cdot}(0,0) + \partial v_2(0,0)/\partial\tau] + \ldots = \beta \quad (4)$$

It therefore follows that

$$u_0(0) = \alpha,\ u_0^{\cdot}(0) + \partial v_1(0,0)/\partial\tau = \beta \quad (5)$$

while the coefficients for $\omega^{-1}, \omega^{-2}, \ldots$ in expansions (4) are zero. Now substituting (3) into (1), we obtain the identity

$$A(q^*)q^{*\cdot\cdot} + B(q^*)q^{*\cdot} = F(t,q) + \omega\Phi(t,q^*,\tau) \quad (6)$$

We try to select u_i and v_i such that the identity (6) would be satisfied for all $x \in [0,T]$ and $\tau \in [0,\infty)$. We equate the coefficients of powers of $\omega, \omega^0, \omega^{-1}, \omega^{-2}$, etc.

We obtain for the first power in ω

$$A^\circ\partial^2 v_1/\partial\tau^2 \equiv \Phi^\circ \quad (\Phi^\circ = \Phi(t, u_0, \tau))$$

Since the matrix A° is reversible and Φ° is a trigonometric polynomial in τ with zero mean, then v_1 can be determined uniquely in the form of a trigonometric polynomial in τ with coefficients dependent on t and u_0

$$v_1 = f_1(t, u_0, \tau) \quad (7)$$

If it is taken into account that $u_0(0) = \alpha$, then the quantity

$$\Psi_1(\alpha) = \partial v_1(0,0)/\partial\tau = \partial f_1(0,\alpha,0)/\partial\tau$$

is completely determined. Consequently $u_0^{\cdot}(0) = \beta - \Psi_1(\alpha)$ is also determined from (5).

Now equating coefficients for the zeroth power of ω in the identity (6), we obtain

$$A^\circ[u_0^{\cdot\cdot} + 2\partial v_1^{\cdot}/\partial\tau + \partial^2 v_1/\partial\tau^2] + A_q^\circ(u_1 + v_1)\partial^2 v_1/\partial\tau^2 +$$
$$B^\circ(u_0^{\cdot} + \partial v_1/\partial\tau) = F^\circ + \Phi_q^0(u_1 + v_1)\ (F^\circ = F(t, u_0)) \quad (8)$$

Let us calculate the mean value of the left and right sides of the last equality with respect to τ. We have (W° is the vibrational force)

$$A^\circ u_0^{\cdot\cdot} + B^\circ u_0^{\cdot} \equiv F^\circ + W^\circ \quad (9)$$
$$W^\circ = \langle\Phi_q^\circ v_1(t,\tau)\rangle - \langle[A_q^\circ v_1(t,\tau)]\,\partial^2 v_1/\partial\tau^2\rangle$$

Therefore, we obtain (9) and the initial conditions $u_0(0) = \alpha$, $u_0^{\cdot}(0) = \beta - \Psi_1(\alpha)$ to determine $u_0(t)$. We assume this problem to be solved in the segment $[0,T]$. Then we finally also obtain $v_1(t,\tau)$ from (7).

We now examine components with zero mean in τ in (8). We obtain

$$A^\circ\partial^2 v_2/\partial\tau^2 = Q_1^\circ + Q_2^\circ u_1 \quad (10)$$

where Q_1° and Q_2° are known vector functions and matrices dependent on t, u_0, τ. We will seek the function v_2 in the form

$$v_2 = w_2(t,\tau) + Z_2(t,\tau)u_1(t)$$

where $w_2(t,\tau)$ is a vector function while $Z_2(t,\tau)$ is a matrix whose coefficients are trigonometric polynomials in τ having zero mean value. Now w_2 and Z_2 are determined uniquely from (10), but, the function v_2 still remains undetermined since the function $u_1(t)$ is undetermined.

We now equate the coefficients for ω^{-1} in (6) and in the equality obtained we take the average with respect to τ. Consequently we obtain the following equation for u_1

$$A^\circ u_1^{\cdot\cdot} + B^\circ u_1^{\cdot} = d_1(t) + C_1(t)u_1 \quad (11)$$

where the vector function $d_1(t)$ and the matrix $C_1(t)$ are determined just by using the u_0, v_1, w_2 and Z_2 **already** known. Let us note that we conclude from the expression in square brackets in (4) being equal to zero that the quantities

$$u_1(0) = -v_1(0,0),\ u_1^{\cdot}(0) = -v_1^{\cdot}(0,0) - \partial v_2(0,0)/\partial\tau \quad (12)$$

are known. This enables u_1 to be determined completely from (11) and (12).

Furthermore, v_3 should be sought in the form

$$v_3 = w_3(t,\tau) + Z_3(t,\tau)u_2(t)$$

etc. This procedure enables us to determine the approximate solution

$$q_N^* = u_0(t) + \omega^{-1}[u_1(t) + v_1(t,\tau)] + \ldots + \omega^{-N}[u_N(t) + v_N(t,\tau)] \quad (13)$$

to any accuracy for any integer $N \geqslant 1$.

If the rapidly oscillating forces are not large and (1) has the form

$$A(q)q^{\cdot\cdot} + B(q)q^{\cdot} = F(t,q) + \Phi(t,q,\tau)$$

then the approximate solution is found in the form (13) where $v_1(t,\tau) \equiv 0$.

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ON THE MITCHELL PROBLEM OF THE MOTION OF A LUBRICANT IN A LAYER BOUNDED BY A MOVING PLANE AND A FIXED PLATE OF FINITE SIZES*

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The Reynolds equation which appears in the hydrodynamic theory of lubrication is applied to the case of the flow of a lubricant between a plane and an inclined plate, and is solved with help of the special functions for a rectangular as well as segmented form of the plate.

1. An unbounded plane moves longitudinally with velocity U in the direction of the x axis. We direct the y axis towards the liquid. Let h be the thickness of the layer, depending only on the coordinate x, $q = h_2/h_1 > 1$ the ratio of the thicknesses of the layer at the plate edges along the x axis, a the distance between these edges and $2l$ the width of the plate in the direction of the z axis.

Using the well-known approximate Reynolds equation, we arrive at the following boundary value problem for the pressure:

$$\frac{\partial}{\partial x}\left(h^3\,\frac{\partial p}{\partial x}\right)+h^3\frac{\partial^2 p}{\partial z^2}=6\mu U\,\frac{\partial h}{\partial x} \tag{1.1}$$

$$\begin{gathered}-l<z<l,\ x=0,\ p=p_a;\ x=a,\ p=p_a\\ 0<x<a,\ z=\pm l,\ p=p_a\end{gathered} \tag{1.2}$$

Since h depends only on x, it follows that a particular solution of Eq.(1.1) can be taken in the form

$$p_0=\chi_0(x)=6\mu U\int h^{-2}dx+C_1\int h^{-3}dx+C_2 \tag{1.3}$$

We shall construct the solution of the corresponding homogeneous equation in the form $p_n = \operatorname{ch}(nz)\,\chi_n(x)$. In this case we obtain the following equation for χ_n:

$$\frac{d}{dx}\left(h^3\,\frac{d\chi_n}{dx}\right)+n^2h^3\chi_n=0 \tag{1.4}$$

whose complete solution will consist of two independent solutions $\chi_n^{(1)}$ and $\chi_n^{(2)}$, so that $\chi_n = A_n\chi_n^{(1)} + B_n\chi_n^{(2)}$.

We can show in the usual manner that the functions χ_n are orthogonal

$$\int_0^a \chi_m\chi_n h^3dx=0,\quad m\neq n \tag{1.5}$$

when the following conditions hold:

$$A_n\chi_n^{(1)}(0)+B_n\chi_n^{(2)}(0)=0,\quad A_n\chi_n^{(1)}(a)+B_n\chi_n^{(2)}(a)=0 \tag{1.6}$$

Combining the particular solution (1.3) with the set of solutions χ_n, we obtain the general solution of Eq.(1.1) in the form

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